

Octahedral, dicyclic and special linear solutions of some unsolved Hamilton-Waterloo problems

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Abstract

We give a sharply-vertex-transitive solution of each of the nine Hamilton-Waterloo problems left open by Danziger, Quattrocchi and Stevens.

1 Introduction

In a long paper still in preparation [1], we will give a general method to obtain decompositions and/or factorizations of a graph with a nice automorphism group, in particular with an automorphism group G acting sharply transitively (i.e., regularly) on the vertices. In this case one also say that the decomposition or factorization is G -regular.

Sharply-vertex-transitive 2-factorizations of a complete graph of odd order have been already partly investigated in [6]. Sharply-vertex-transitive 2-factorizations of a cocktail party graph (that is a complete graph of even order minus a 1-factor) can be treated similarly. Here we show how the method of *partial differences* explained in [5] and successfully applied to solve several cycle decompositions problems (see, e.g., [4]) allows to obtain a sharply-vertex-transitive solution of each of the nine Hamilton-Waterloo problems left open by Danziger, Quattrocchi and Stevens [7]. We namely prove that: 1) there exists a regular 2-factorization of $K_{48} - I$ having exactly r triangle-factors with $r \in \{5, 7, 9, 13, 15, 17\}$ and each of the remaining factors consisting of all quadrangles; 2) there exists a regular 2-factorization of $K_{24} - I$ having exactly r triangle-factors with $r \in \{5, 7, 9\}$ and each of the remaining factors consisting of all quadrangles. The choice of the group acting regularly on our factorizations cannot be random; in each case we have been forced to take an ad hoc non-abelian group with exactly one involution.

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2 Octahedral solutions of six Hamilton-Waterloo problems

Throughout this section G will denote the so-called *binary octahedral group* which is usually denoted by \overline{O} . This group, up to isomorphism, can be viewed as a group of units of the skew-field \mathbb{H} of *quaternions* introduced by Hamilton, that is an extension of the complex field \mathbb{C} . We recall the basic facts regarding \mathbb{H} . Its elements are all real linear combinations of $1, i, j$ and k . The sum and the product of two quaternions are defined in the natural way under the rules that

$$i^2 = j^2 = k^2 = ijk = -1.$$

If $q = a + bi + cj + dk \neq 0$, its inverse is given by

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.$$

The elements of the multiplicative group G can be described as follows:

$$\begin{aligned} & \pm 1, \pm i, \pm j, \pm k; \\ & \frac{1}{2}(\pm 1 \pm i \pm j \pm k); \\ & \frac{1}{\sqrt{2}}(\pm x \pm y), \quad \{x, y\} \in \binom{\{1, i, j, k\}}{2}. \end{aligned}$$

The use of octahedral group G was crucial in [2] to get a Steiner triple system of any order $v = 96n + 49$ with an automorphism group acting sharply transitively on all but one point. Here G will be used to get a G -regular solution of each of the six Hamilton-Waterloo problems of order 48 left open in [7]. We will need to consider the following subgroups of G order 16 and 12, respectively:

- $K = \langle k, \frac{1}{\sqrt{2}}(j - k) \rangle;$
- $L = \langle \frac{1}{\sqrt{2}}(j - k), \frac{1}{2}(-1 - i + j + k) \rangle.$

In the following the complete graph K_{48} and the cocktail party graph $K_{48} - I$ will be seen as the Cayley graphs $\text{Cay}[G : G \setminus \{1\}]$ and $\text{Cay}[G : G \setminus \{1, -1\}]$, respectively.

2.1 An octahedral solution of HWP(48; 3, 4; 5, 18)

Consider the nine cycles of K_{48} defined as follows.

$$\begin{aligned}
C_1 &= (1, -\frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i-j-k)) \\
C_2 &= (1, \frac{1}{2}(-1-i+j+k), \frac{1}{2}(-1+i-j-k)) \\
C_3 &= (1, \frac{1}{2}(-1+i+j-k), \frac{1}{2}(-1-i-j+k)) \\
C_4 &= (1, k, -1, -k) \\
C_5 &= (1, j, -1, -j) \\
C_6 &= (1, \frac{1}{\sqrt{2}}(-i+k), -\frac{1}{2}(1+i+j+k), -\frac{1}{\sqrt{2}}(j+k)) \\
C_7 &= (1, \frac{1}{\sqrt{2}}(i-j), \frac{1}{\sqrt{2}}(1+i), \frac{1}{2}(1-i-j+k)) \\
C_8 &= (1, \frac{1}{2}(1-i+j-k), k, -\frac{1}{\sqrt{2}}(1+j)) \\
C_9 &= (1, \frac{1}{\sqrt{2}}(1-i), -\frac{1}{\sqrt{2}}(1+i), \frac{1}{2}(-1-i+j-k))
\end{aligned}$$

We note that for $2 \leq i \leq 5$ the G -stabilizer of C_i is $V(C_i)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $Orb_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \{-\frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i-j-k), -\frac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\
\Omega_2 &= \{\frac{1}{2}(-1-i+j+k)\}^{\pm 1} \\
\Omega_3 &= \{\frac{1}{2}(-1+i+j-k)\}^{\pm 1} \\
\Omega_4 &= \{k\}^{\pm 1} \\
\Omega_5 &= \{j\}^{\pm 1} \\
\Omega_6 &= \{\frac{1}{\sqrt{2}}(-i+k), \frac{1}{\sqrt{2}}(j-k), \frac{1}{\sqrt{2}}(1-k), -\frac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\
\Omega_7 &= \{\frac{1}{\sqrt{2}}(i-j), \frac{1}{2}(1+i-j-k), \frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i-j+k)\}^{\pm 1} \\
\Omega_8 &= \{\frac{1}{2}(1-i+j-k), -\frac{1}{2}(1+i+j+k), -\frac{1}{\sqrt{2}}(i+k), -\frac{1}{\sqrt{2}}(1+j)\}^{\pm 1} \\
\Omega_9 &= \{\frac{1}{\sqrt{2}}(1-i), i, \frac{1}{\sqrt{2}}(1+j), \frac{1}{2}(-1-i+j-k)\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i 's partition $G \setminus \{1, -1\}$. This assures that

$$\mathcal{C} := \bigcup_{i=1}^7 Orb_G(C_i) \text{ is a } G\text{-regular cycle-decomposition of } K_{48} - I.$$

$$\text{Now set } F_i = Orb_{S_i}(C_i) \text{ where } S_i = \begin{cases} K & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ L & \text{for } 6 \leq i \leq 9. \end{cases}$$

Each F_i is a 2-factor of K_{48} with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1$, or $2 \leq i \leq 5$, or $6 \leq i \leq 9$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that \mathcal{C} is a cycle-decomposition of

$$K_{48} - I, \text{ we conclude that } \mathcal{F} := \bigcup_{i=1}^9 Orb_G(F_i) \text{ is a } G\text{-regular 2-factorization}$$

of $K_{48} - I$ with 5 triangle-factors and 18 quadrangle-factors, namely a G -regular solution of $\text{HWP}(48; 3, 4; 5, 18)$.

2.2 An octahedral solution of $\text{HWP}(48; 3, 4; 7, 16)$

Consider the seven cycles of K_{48} defined as follows.

$$\begin{aligned} C_1 &= (1, -\frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i+j+k)) \\ C_2 &= (1, \frac{1}{2}(-1-i+j+k), \frac{1}{2}(1-i-j-k)) \\ C_3 &= (1, \frac{1}{2}(-1+i+j-k), \frac{1}{2}(-1-i-j+k)) \\ C_4 &= (1, \frac{1}{\sqrt{2}}(-i+k), \frac{1}{2}(1+i+j-k), -\frac{1}{\sqrt{2}}(j+k)) \\ C_5 &= (1, \frac{1}{\sqrt{2}}(i-j), \frac{1}{\sqrt{2}}(1-k), \frac{1}{\sqrt{2}}(1+i)) \\ C_6 &= (1, \frac{1}{\sqrt{2}}(1+k), -\frac{1}{2}(1+i+j+k), \frac{1}{\sqrt{2}}(1+j)) \\ C_7 &= (1, -\frac{1}{2}(1+i+j+k), \frac{1}{2}(1-i+j-k), \frac{1}{2}(1-i-j+k)) \end{aligned}$$

We note that the G -stabilizer of C_3 is $V(C_3)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{-\frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i+j+k), \frac{1}{\sqrt{2}}(-j+k)\}^{\pm 1} \\ \Omega_2 &= \{\frac{1}{2}(-1-i+j+k), \frac{1}{2}(1-i-j-k), \frac{1}{2}(-1-i+j-k)\}^{\pm 1} \\ \Omega_3 &= \{\frac{1}{2}(-1+i+j-k)\}^{\pm 1} \\ \Omega_4 &= \{\frac{1}{\sqrt{2}}(-i+k), -\frac{1}{\sqrt{2}}(1-k), \frac{1}{\sqrt{2}}(i+k), -\frac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\ \Omega_5 &= \{\frac{1}{\sqrt{2}}(i-j), -j, \frac{1}{2}(1-i+j-k), \frac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\ \Omega_6 &= \{\frac{1}{\sqrt{2}}(1+k), \frac{1}{\sqrt{2}}(-1+j), -\frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(1+j)\}^{\pm 1} \\ \Omega_7 &= \{-\frac{1}{2}(1+i+j+k), -i, -k, \frac{1}{2}(1-i-j+k)\}^{\pm 1} \end{aligned}$$

One can see that the Ω_i 's partition $G \setminus \{1, -1\}$. This assures that $\mathcal{C} := \bigcup_{i=1}^7 \text{Orb}_G(C_i)$ is an G -regular cycle-decomposition of $K_{48} - I$.

$$\text{Now set } F_i = \text{Orb}_{S_i}(C_i) \text{ where } S_i = \begin{cases} K & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ L & \text{for } 4 \leq i \leq 7. \end{cases}$$

Each F_i is a 2-factor of K_{48} with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $4 \leq i \leq 7$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we conclude

that $\mathcal{F} := \bigcup_{i=1}^7 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{48} - I$ with 7 triangle-factors and 16 quadrangle-factors, namely a G -regular solution of HWP(48; 3, 4; 7, 16).

2.3 An octahedral solution of HWP(48; 3, 4; 9, 14)

Consider the eight cycles of K_{48} defined as follows.

$$\begin{aligned} C_1 &= (1, \frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i-j-k)) \\ C_2 &= (1, -\frac{1}{\sqrt{2}}(1-k), \frac{1}{\sqrt{2}}(1+j)) \\ C_3 &= (1, \frac{1}{2}(-1-i+j+k), \frac{1}{2}(1+i-j+k)) \\ C_4 &= (1, \frac{1}{\sqrt{2}}(-i+k), \frac{1}{\sqrt{2}}(1-i), \frac{1}{2}(-1-i+j-k)) \\ C_5 &= (1, \frac{1}{\sqrt{2}}(i-j), \frac{1}{2}(-1+i+j+k), -\frac{1}{\sqrt{2}}(j+k)) \\ C_6 &= (1, \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(1-i), \frac{1}{2}(1-i-j+k)) \\ C_7 &= (1, k, -1, -k) \\ C_8 &= (1, j, -1, -j) \end{aligned}$$

We note that for $i = 7, 8$ the G -stabilizer of C_i is $V(C_i)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{ \frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i-j-k), \frac{1}{\sqrt{2}}(-1+i) \}^{\pm 1} \\ \Omega_2 &= \{ -\frac{1}{\sqrt{2}}(1-k), \frac{1}{\sqrt{2}}(1+j), \frac{1}{2}(-1+i+j+k) \}^{\pm 1} \\ \Omega_3 &= \{ \frac{1}{2}(-1-i+j+k), \frac{1}{2}(1+i-j+k), \frac{1}{2}(-1-i-j+k) \}^{\pm 1} \\ \Omega_4 &= \{ \frac{1}{\sqrt{2}}(-i+k), \frac{1}{2}(1-i+j+k), \frac{1}{\sqrt{2}}(i+k), \frac{1}{2}(-1-i+j-k) \}^{\pm 1} \\ \Omega_5 &= \{ \frac{1}{\sqrt{2}}(i-j), \frac{1}{\sqrt{2}}(j-k), -\frac{1}{\sqrt{2}}(1+j), -\frac{1}{\sqrt{2}}(j+k) \}^{\pm 1} \\ \Omega_6 &= \{ \frac{1}{\sqrt{2}}(1+i), i, \frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i-j+k) \}^{\pm 1} \\ \Omega_7 &= \{ k \}^{\pm 1} \\ \Omega_8 &= \{ j \}^{\pm 1} \end{aligned}$$

Now note that the Ω_i 's partition $G \setminus \{1, -1\}$. This assures that $\mathcal{C} := \bigcup_{i=1}^8 \text{Orb}_G(C_i)$ is a G -regular cycle-decomposition of $K_{48} - I$.

$$\text{Now set } F_i = \text{Orb}_{S_i}(C_i) \text{ where } S_i = \begin{cases} K & \text{for } 1 \leq i \leq 3; \\ L & \text{for } 4 \leq i \leq 6; \\ G & \text{for } i = 7, 8. \end{cases}$$

Each F_i is a 2-factor of K_{48} with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 4 or 1 according to whether $1 \leq i \leq 3$ or $4 \leq i \leq 6$ or $i = 7, 8$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^8 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{48} - I$ with 9 triangle-factors and 14 quadrangle-factors, namely a G -regular solution of $\text{HWP}(48; 3, 4; 9, 14)$.

2.4 An octahedral solution of $\text{HWP}(48; 3, 4; 13, 10)$

Consider the nine cycles of K_{48} defined as follows.

$$\begin{aligned} C_1 &= (1, -\frac{1}{\sqrt{2}}(i+j), -\frac{1}{\sqrt{2}}(1+j)) \\ C_2 &= (1, \frac{1}{2}(1-i+j-k), -\frac{1}{\sqrt{2}}(i+k)) \\ C_3 &= (1, \frac{1}{\sqrt{2}}(-i+j), \frac{1}{2}(1-i-j-k)) \\ C_4 &= (1, \frac{1}{2}(-1+i-j+k), \frac{1}{\sqrt{2}}(i-k)) \\ C_5 &= (1, \frac{1}{2}(-1-i+j+k), \frac{1}{2}(-1+i-j-k)) \\ C_6 &= (1, -\frac{1}{2}(1+i+j+k), \frac{1}{2}(-1+i-j+k), \frac{1}{\sqrt{2}}(1+j)) \\ C_7 &= (1, -\frac{1}{\sqrt{2}}(1+k), -k, \frac{1}{2}(-1+i+j-k)) \\ C_8 &= (1, k, -1, -k) \\ C_9 &= (1, j, -1, -j) \end{aligned}$$

We note that for $5 \leq i \leq 7$ the G -stabilizer of C_i is $V(C_i)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{-\frac{1}{\sqrt{2}}(i+j), -\frac{1}{\sqrt{2}}(1+j), \frac{1}{2}(1+i+j-k)\}^{\pm 1} \\ \Omega_2 &= \{\frac{1}{2}(1-i+j-k), -\frac{1}{\sqrt{2}}(i+k), \frac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\ \Omega_3 &= \{\frac{1}{\sqrt{2}}(-i+j), \frac{1}{2}(1-i-j-k), \frac{1}{\sqrt{2}}(j-k)\}^{\pm 1} \\ \Omega_4 &= \{\frac{1}{2}(-1+i-j+k), \frac{1}{\sqrt{2}}(i-k), -\frac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\ \Omega_5 &= \{\frac{1}{2}(-1-i+j+k)\}^{\pm 1} \\ \Omega_6 &= \{k\}^{\pm 1} \\ \Omega_7 &= \{j\}^{\pm 1} \\ \Omega_8 &= \{-\frac{1}{2}(1+i+j+k), i, \frac{1}{\sqrt{2}}(-1+i), \frac{1}{\sqrt{2}}(1+j)\}^{\pm 1} \\ \Omega_9 &= \{-\frac{1}{\sqrt{2}}(1+k), \frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i+j+k), \frac{1}{2}(-1+i+j-k)\}^{\pm 1} \end{aligned}$$

Now note that the Ω_i 's partition $G \setminus \{1, -1\}$. This assures that

$\mathcal{C} := \bigcup_{i=1}^9 \text{Orb}_G(C_i)$ is a G -regular cycle-decomposition of $K_{48} - I$.

Now set $F_i = \text{Orb}_{S_i}(C_i)$ where $S_i = \begin{cases} K & \text{for } 1 \leq i \leq 4; \\ G & \text{for } 5 \leq i \leq 7; \\ L & \text{for } i = 8, 9. \end{cases}$

Each F_i is a 2-factor of K_{48} with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \leq i \leq 4$ or $5 \leq i \leq 7$ or $i = 8, 9$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we conclude

that $\mathcal{F} := \bigcup_{i=1}^9 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{48} - I$ with 13 triangle-factors and 10 quadrangle-factors, namely a G -regular solution of $\text{HWP}(48; 3, 4; 13, 10)$.

2.5 An octahedral solution of $\text{HWP}(48; 3, 4; 15, 8)$

Consider the seven cycles of K_{48} defined as follows.

$$\begin{aligned} C_1 &= (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{\sqrt{2}}(i + k)) \\ C_2 &= (1, -\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j)) \\ C_3 &= (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(1 - i + j + k)) \\ C_4 &= (1, \frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j)) \\ C_5 &= (1, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(i - k)) \\ C_6 &= (1, -j, k, -\frac{1}{\sqrt{2}}(1 - k)) \\ C_7 &= (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{2}(-1 - i + j - k), \frac{1}{2}(-1 + i + j + k)) \end{aligned}$$

Here, every C_i has trivial G -stabilizer. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned}
\Omega_1 &= \left\{ \frac{1}{2}(-1 - i + j + k), \frac{1}{\sqrt{2}}(i + k), \frac{1}{\sqrt{2}}(-j + k) \right\}^{\pm 1} \\
\Omega_2 &= \left\{ -\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j), \frac{1}{2}(1 + i + j - k) \right\}^{\pm 1} \\
\Omega_3 &= \left\{ \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(1 - i + j + k), \frac{1}{2}(-1 - i + j - k) \right\}^{\pm 1} \\
\Omega_4 &= \left\{ \frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j), \frac{1}{\sqrt{2}}(1 + i) \right\}^{\pm 1} \\
\Omega_5 &= \left\{ \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(i - k), \frac{1}{\sqrt{2}}(j + k) \right\}^{\pm 1} \\
\Omega_6 &= \left\{ -j, +i, \frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(1 - k) \right\}^{\pm 1} \\
\Omega_7 &= \left\{ \frac{1}{\sqrt{2}}(i - j), -\frac{1}{\sqrt{2}}(1 + i), +k, \frac{1}{2}(-1 + i + j + k) \right\}^{\pm 1}
\end{aligned}$$

Now note that the Ω_i 's partition $G \setminus \{1, -1\}$. This assures that

$$\mathcal{C} := \bigcup_{i=1}^8 \text{Orb}_G(C_i) \text{ is a } G\text{-regular cycle-decomposition of } K_{48} - I.$$

$$\text{Set } F_i = \text{Orb}_{S_i}(C_i) \text{ where } S_i = \begin{cases} K & \text{for } 1 \leq i \leq 5; \\ L & \text{for } i = 6, 7. \end{cases}$$

Each F_i is a 2-factor of K_{48} with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 4 according to whether $1 \leq i \leq 5$ or $i = 6, 7$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we conclude

that $\mathcal{F} := \bigcup_{i=1}^7 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{48} - I$ with 15 triangle-factors and 8 quadrangle-factors, namely a G -regular solution of HWP(48; 3, 4; 15, 8).

2.6 An octahedral solution of HWP(48; 3, 4; 17, 6)

Consider the ten cycles of K_{48} defined as follows.

$$\begin{aligned}
C_1 &= (1, -\frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(i + k)) \\
C_2 &= (1, -\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(-1 + i + j + k)) \\
C_3 &= (1, \frac{1}{2}(1 + i - j - k), -\frac{1}{\sqrt{2}}(1 + j)) \\
C_4 &= (1, \frac{1}{\sqrt{2}}(-i + j), \frac{1}{\sqrt{2}}(-i + k)) \\
C_5 &= (1, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 - j)) \\
C_6 &= (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 + i - j - k)) \\
C_7 &= (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 - i - j + k)) \\
C_8 &= (1, k, -1, -k) \\
C_9 &= (1, j, -1, -j) \\
C_{10} &= (1, \frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 - i), \frac{1}{2}(1 - i - j + k))
\end{aligned}$$

We note that for $6 \leq i \leq 9$ the G -stabilizer of C_i is $V(C_i)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $Orb_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned}\Omega_1 &= \{-\frac{1}{\sqrt{2}}(1-k), -\frac{1}{\sqrt{2}}(i+k), \frac{1}{2}(-1-i+j-k)\}^{\pm 1} \\ \Omega_2 &= \{-\frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(-1+i+j+k), \frac{1}{\sqrt{2}}(-1+i)\}^{\pm 1} \\ \Omega_3 &= \{\frac{1}{2}(1+i-j-k), -\frac{1}{\sqrt{2}}(1+j), \frac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\ \Omega_4 &= \{\frac{1}{\sqrt{2}}(-i+j), \frac{1}{\sqrt{2}}(-i+k), \frac{1}{2}(1-i-j-k)\}^{\pm 1} \\ \Omega_5 &= \{\frac{1}{2}(1-i+j-k), \frac{1}{\sqrt{2}}(1-j), \frac{1}{\sqrt{2}}(j-k)\}^{\pm 1} \\ \Omega_6 &= \{\frac{1}{2}(-1-i+j+k)\}^{\pm 1} \\ \Omega_7 &= \{\frac{1}{2}(-1+i+j-k)\}^{\pm 1} \\ \Omega_8 &= \{k\}^{\pm 1} \\ \Omega_9 &= \{j\}^{\pm 1} \\ \Omega_{10} &= \{\frac{1}{\sqrt{2}}(1+i), i, \frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i-j+k)\}^{\pm 1}\end{aligned}$$

Now note that the Ω_i 's partition $G \setminus \{1, -1\}$. This assures that $\mathcal{C} := \bigcup_{i=1}^8 Orb_G(C_i)$ is a G -regular cycle-decomposition of $K_{48} - I$.

$$\text{Set } F_i = Orb_{S_i}(C_i) \text{ where } S_i = \begin{cases} K & \text{for } 1 \leq i \leq 5; \\ G & \text{for } 6 \leq i \leq 9; \\ L & \text{for } i = 10. \end{cases}$$

Each F_i is a 2-factor of K_{48} with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \leq i \leq 5$ or $6 \leq i \leq 9$ or $i = 10$, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 7$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{10} Orb_G(F_i)$ is a G -regular 2-factorization of $K_{48} - I$ with 17 triangle-factors and 6 quadrangle-factors, namely a G -regular solution of $\text{HWP}(48; 3, 4; 17, 6)$.

3 Dicyclic solutions of two Hamilton-Waterloo problems

In this section G will denote the dicyclic group of order 24 which is usually denoted by Q_{24} . Thus G has the following presentation:

$$G = \langle a, b \mid a^{12} = 1, b^2 = a^6, b^{-1}ab = a^{-1} \rangle$$

Note that the elements of G can be written in the form $a^i b^j$ with $0 \leq i \leq 11$ and $j = 0, 1$. The group G has a unique involution which is a^6 and we will need to consider the following subgroups of G :

- $H = \langle b \rangle = \{1, b, a^6, a^6 b\}$;
- $K = \langle a^2 \rangle = \{1, a^2, a^4, a^6, a^8, a^{10}\}$;
- $L = \langle a^2 b, a^3 \rangle = \{1, a^3, a^6, a^9, a^2 b, a^8 b, a^5 b, a^{11} b\}$.

In the following the complete graph K_{24} and the cocktail party graph $K_{24} - I$ will be seen as the Cayley graphs $\text{Cay}[G : G \setminus \{1\}]$ and $\text{Cay}[G : G \setminus \{1, a^6\}]$, respectively.

3.1 A dicyclic solution of HWP(24; 3, 4; 7, 4)

Consider the four cycles of K_{24} defined as follows.

$$\begin{aligned} C_1 &= (1, a^3 b, a^5) \\ C_2 &= (1, a^{10}, a^7 b) \\ C_3 &= (1, a^4, a^8) \\ C_4 &= (1, b, a^3 b, a) \end{aligned}$$

We note that the G -stabilizer of C_3 is $V(C_3)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{a^3 b, a^5, a^2 b\}^{\pm 1} \\ \Omega_2 &= \{a^2, ab, a^5 b\}^{\pm 1} \\ \Omega_3 &= \{a^4\}^{\pm 1} \\ \Omega_4 &= \{b, a^3, a^4 b, a\}^{\pm 1} \end{aligned}$$

Now note that the Ω_i 's partition $G \setminus \{1, a^6\}$. This assures that $\mathcal{C} := \bigcup_{i=1}^4 \text{Orb}_G(C_i)$ is a G -regular cycle-decomposition of $K_{24} - I$.

$$\text{Now set } F_i = \text{Orb}_{S_i}(C_i) \text{ where } S_i = \begin{cases} L & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ K & \text{for } i = 4. \end{cases}$$

Each F_i is a 2-factor of K_{24} with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $i = 4$, respectively.

The cycles of F_i are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we

conclude that $\mathcal{F} := \bigcup_{i=1}^4 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{24} - I$ with 7 triangle-factors and 4 quadrangle-factors, namely a G -regular solution of $\text{HWP}(24; 3, 4; 7, 4)$.

3.2 A dicyclic solution of $\text{HWP}(24; 3, 4; 9, 2)$

Consider the four cycles of K_{24} defined as follows.

$$\begin{aligned} C_1 &= (1, b, a^6, a^6b) \\ C_2 &= (1, a^4b, a^6, a^{10}b) \\ C_3 &= (1, a^4, a^7b) \\ C_4 &= (1, a^3b, a^8b) \\ C_5 &= (a^4, a^7, a^5) \end{aligned}$$

We note that for $i = 1, 2$ the G -stabilizer of C_i is $V(C_i)$ while all other C_i 's have trivial G -stabilizer. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$ is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned} \Omega_1 &= \{b\}^{\pm 1} \\ \Omega_2 &= \{a^4b\}^{\pm 1} \\ \Omega_3 &= \{a^4, ab, a^5b\}^{\pm 1} \\ \Omega_4 &= \{a^3b, a^2b, a^5\}^{\pm 1} \\ \Omega_5 &= \{a^1, a^2, a^3\}^{\pm 1} \end{aligned}$$

Also here the Ω_i 's partition $G \setminus \{1, a^6\}$, hence $\mathcal{C} := \bigcup_{i=1}^5 \text{Orb}_G(C_i)$ is a G -regular cycle-decomposition of $K_{24} - I$. Now set:

$$\begin{aligned} F_1 &= \text{Orb}_G(C_1), & F_2 &= \text{Orb}_G(C_2), \\ F_3 &= \text{Orb}_L(C_3), & F_4 &= \text{Orb}_H(C_4) \cup \text{Orb}_H(C_5). \end{aligned}$$

Each F_i is a 2-factor of K_{24} and we have

$$\text{Stab}_G(F_1) = \text{Stab}_G(F_2) = G; \quad \text{Stab}_G(F_3) = L; \quad \text{Stab}_G(F_4) = H$$

so that the lengths of the G -orbits of F_1, \dots, F_4 are 1, 1, 3 and 6, respectively. The cycles of F_i are triangles or quadrangles according to whether or not $i \geq 3$. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^5 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{24} - I$ with

9 triangle-factors and 2 quadrangle-factors, namely a G -regular solution of $\text{HWP}(24; 3, 4; 9, 2)$.

4 A special linear solution of HWP(24; 3, 4; 5, 6)

In this section G will denote the 2-dimensional special linear group over \mathbb{Z}_3 , usually denoted by $SL_2(3)$, namely the group of 2×2 matrices with elements in \mathbb{Z}_3 and determinant one. The only involution of G is $2E$ where E is the identity matrix of G . The 2-Sylow subgroup Q of G , isomorphic to the group of quaternions, is the following:

$$Q = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

We will also need to consider the subgroup H of G of order 6 generated by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Hence we have:

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right\}.$$

The use of the special linear group G was crucial in [5] to get a Steiner triple system of any order $v = 144n + 25$ with an automorphism group acting sharply transitively on all but one point. Here G will be used to get a G -regular solution of the last Hamilton-Waterloo problem left open in [7]. In the following the complete graph K_{24} and the cocktail party graph $K_{24} - I$ will be seen as the Cayley graphs $\text{Cay}[G : G \setminus \{E\}]$ and $\text{Cay}[G : G \setminus \{E, -E\}]$, respectively.

Consider the six cycles of K_{24} defined as follows.

$$\begin{aligned} C_1 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right) \\ C_2 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \right) \\ C_3 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \right) \\ C_4 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \\ C_5 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \right) \\ C_6 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \end{aligned}$$

Here the G -stabilizer of C_i is trivial for $i = 1, 6$ while it coincides with $V(C_i)$ for $2 \leq i \leq 5$. Thus, using partial differences, one can check that $\text{Orb}_G(C_i)$

is a ℓ_i -cycle decomposition of $\text{Cay}[G : \Omega_i]$ where ℓ_i is the length of C_i and where the Ω_i 's are the symmetric subsets of G listed below.

$$\begin{aligned}\Omega_1 &= \left\{ \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}^{\pm 1} \\ \Omega_2 &= \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1} & \Omega_3 &= \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \right\}^{\pm 1} \\ \Omega_4 &= \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}^{\pm 1} & \Omega_5 &= \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1} \\ \Omega_6 &= \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}^{\pm 1}\end{aligned}$$

Once again we see that the Ω_i 's partition $G \setminus \{E, 2E\}$, therefore $\mathcal{C} := \bigcup_{i=1}^4 \text{Orb}_G(C_i)$ is a G -regular cycle-decomposition of $K_{24} - I$.

Now set $F_i = \text{Orb}_{S_i}(C_i)$ with $S_i = \begin{cases} Q & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ H & \text{for } i = 6. \end{cases}$

Each F_i is a 2-factor of K_{24} and we have $\text{Stab}_G(F_i) = S_i$ so that the lengths of the G -orbits of F_1, \dots, F_6 are 3, 1, 1, 1, 1 and 4, respectively.

The cycles of F_i have length 3 or 4 according to whether or not $i \leq 3$ or not. Thus, recalling that \mathcal{C} is a cycle-decomposition of $K_{24} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^6 \text{Orb}_G(F_i)$ is a G -regular 2-factorization of $K_{24} - I$ with 5 triangle-factors and 6 quadrangle-factors, namely a G -regular solution of $\text{HWP}(24; 3, 4; 5, 5)$.

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